Kuwait University | Math 102 | Date: Oct. 31st, 2010

Department of Mathematics | First Exam | Duration: 90 min.

Calculators, mobile phones, pagers and all other mobile communication equipment are not allowed. You must show your work and all important steps to receive credit.

Answer the following questions:

1. Let [2+3+2 pts.]

$$f(x) = \left(2 + \sin^{-1}\left(\frac{x}{3}\right)\right)^3, \qquad x \in [0,3].$$

a. Show that f^{-1} exists.

b. Find f^{-1} and its domain.

c. Find the equation of the tangent line to the curve $y = f^{-1}(x)$ at $x = 8$.

2. Prove the following identities: [2+2 pts.]

a. $e^x e^y = e^{x+y}$.

b. $\log_x(xy) \cdot \log_y(xy) = \log_x y + \log_y x + 2$ for any positive values $x, y \neq 1$.

3. Solve the following inequality for x: [2 pts.]

$$\tan^{-1}[2^x - 5] \geq 0$$

4. Find the exact value of: [2 pts.]

$$\cos\left(\cot^{-1}\left(\frac{4}{3}\right)\right).$$

5. Evaluate each of the following integrals: [3+3pts.]

a. $\displaystyle\int \frac{dx}{\sqrt{4^x - 1}}$.

b. $\displaystyle\int \frac{\sinh x}{e^{\cosh x} + e^{-\cosh x}}\, dx$.

6. Use logarithmic differentiation to find $\frac{dy}{dx}$ if [4 pts.]

$$y = \frac{e^{x^2}\cos^{-1}(x)}{(\sinh x)^{2x}}$$

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Department of Mathematics First Exam Solution

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1. **a.** $f'(x) = \dfrac{\left(2+\sin^{-1}\left(\frac{x}{3}\right)\right)^2}{\sqrt{1-\left(\frac{x^2}{9}\right)}} > 0$, therefore f is increasing in $[0,3)$ $\Rightarrow$ f is 1-1 and f^{-1} exists.

b. $y = \left(2+\sin^{-1}\left(\frac{x}{3}\right)\right)^3 \Rightarrow y^{1/3}-2 = \sin^{-1}\left(\frac{x}{3}\right) \Rightarrow x = 3\sin(y^{1/3}-2)$ or $f^{-1}(x) = 3\sin(x^{1/3}-2)$. Domain of f^{-1} is $\left[8, \left(2+\frac{\pi}{2}\right)^3\right]$, where $\lim_{x\to 3^-} f(x) = \left(2+\frac{\pi}{2}\right)^3$ and $\lim_{x\to 0^+} f(x) = 2^3$.

c. $\dfrac{d}{dx}f^{-1}(8) = \dfrac{1}{f'\left(f^{-1}(8)\right)} = \dfrac{1}{f'(0)} = \dfrac{1}{4}$. The equation of the tangent line is $y = \dfrac{x}{4} - 2$.

2. **a.** In the book.

b. $\log_x(xy)\cdot\log_y(xy) = \dfrac{(\ln x+\ln y)^2}{\ln x \ln y} = \dfrac{(\ln x)^2+(\ln y)^2+2\ln x\ln y}{\ln x\ln y} = \dfrac{\ln x}{\ln y}+\dfrac{\ln y}{\ln x}+2 = \log$

3. $\tan^{-1}[2^x-5] \ge 0 \Rightarrow 2^x-5 \ge \tan 0 \Rightarrow 2^x \ge 5 \Rightarrow x \ge \log_2 5$.

4. Let $\theta = \cot^{-1}\left(\frac{4}{3}\right)$, then $\cot\theta = \frac{4}{3}$. Therefore, $\cos\theta = \dfrac{4}{\sqrt{4^2+3^2}} = \dfrac{4}{5}$.

5. **a.** Let $u = 2^x \Rightarrow du = 2^x \ln 2\, dx$. Then,

$$\int \frac{2^x dx}{2^x\sqrt{(2^x)^2-1}} = \frac{1}{\ln 2}\int \frac{du}{u\sqrt{u^2-1}} = \frac{1}{\ln 2}\sec^{-1}u + c.$$

b. Let $u = \cosh x \Rightarrow du = \sinh x\, dx$. Then, $\displaystyle\int \frac{\sinh x\, dx}{e^{\cosh x}+e^{-\cosh x}} = \int \frac{du}{e^u+e^{-u}}$. Let $z = e^u \Rightarrow dz = e^u du$, then $\displaystyle\int \frac{dz}{z^2+1} = \tan^{-1}(e^{\cosh x}) + c$. (Or let $t = e^{\cosh x} \Rightarrow \dfrac{dt}{t} = \sinh x\, dx$.)

6. $\ln y = \ln\left(e^{x^2}\cos^{-1}(x)\right) - 2^x\ln(\sinh x) = x^2 + \ln\cos^{-1}(x) - 2^x\ln(\sinh x)$.

Differentiating $\Rightarrow \dfrac{1}{y}y' = \left[2x - \dfrac{(1-x^2)^{-1/2}}{\cos^{-1}(x)}\right] - [2^x(\ln 2)\ln(\sinh x) + 2^x\coth x]$.

Therefore, $\dfrac{dy}{dx} = y\left[2x - \dfrac{(1-x^2)^{-1/2}}{\cos^{-1}(x)} - 2^x(\ln 2)\ln(\sinh x) - 2^x\coth x\right]$.